J. Fluid Mech. (1986), *vol.* 167, *pp.* 117–130
Printed in Great Britain

On hydromagnetic critical layers

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(Received 7 February 1985 and in revised form 5 December 1985)

When a horizontal magnetic field $B(z)$ is sheared vertically on a lengthscale L in a diffusionless fluid, critical layers occur at z_c where the local Alfvén speed $V(z_c)$ matches the phase speed c of the wave. However, when a vertical field B_z is introduced, all the critical layers disappear. The present study investigates the solution in the neighbourhood of z_c when B_z/B is very small, in order to clarify the manner in which the vertical field annihilates the critical layers. It is found that the solution across the critical layer is adjusted in a thin magnetic layer whose thickness is determined by the parameter ϵ^2 ($= U/\alpha V$, where U, V are measures of the vertical and horizontal components of the Alfvén velocity and α/L is the horizontal wavenumber). The vertical field increases the order of the equation governing the vertical variations of the amplitude of the perturbations from two to four. Within the magnetic layer the two extra Alfvén waves, one upgoing and the other downgoing, interact with those due to the horizontal field to make the solution regular everywhere. The mean vertical wave energy flux varies continuously from one constant value far on one side of the layer to another constant value far on the other side of the layer.

The influence of the vertical field on the resistive instabilities present in its absence is also examined. It is found that the relative importance of resistivity and vertical field is measured by the ratio of the thicknesses of the resistive and magnetic layers. In general, the influence of the vertical field is to suppress resistive instabilities. The slow exchange resistive instabilities are suppressed by the presence of the vertical field if $\epsilon \geqslant a(S\alpha)^{-\frac{1}{3}}$ while the localized gravitational modes are inhibited for $\epsilon \geqslant b(\alpha^2 S)^{-\frac{1}{4}}$, where a, b are constants whose values depend on the profile of the horizontal field and on the gravitational parameter G; and S is the Lundquist number.

1. Introduction

The propagation and stability properties of hydromagnetic waves in dissipationless fluids under a variety of constraints are relevant to a wide spectrum of applications ranging from geophysical phenomena (Hide 1966; Braginskii 1967; Roberts & Soward 1972; Moffatt 1978; Eltayeb 1981*a*) to astrophysical problems (Lehnert 1954; Moffatt 1973; Acheson 1978) to the construction of thermonuclear reactors (e.g. Bateman 1978).

In a previous paper (Eltayeb 1981*b*) the stability of a horizontal layer, of thickness L, permeated by a horizontal magnetic field sheared in the vertical direction was

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studied. It was shown that the existence of a critical level, i.e. a level z_c at which the phase speed of the wave matches the local Alfvén speed, gives rise to instability. The critical level is manifest in the form of a regular singularity, at z_c, of the second-order ordinary differential equation governing the variation of the amplitude of the wave in the vertical direction. El Mekki, Eltayeb & McKenzie (1978) studied the critical levels of magnetoacoustic waves in the solar atmosphere and indicated that they lead to energy absorption or emission. A recent study by Schwartz & Bel (1984) on wave propagation in a fluid sheared vertically in the presence of a magnetic field slightly inclined to the horizontal showed that the solutions indicated no critical level behaviour and concluded that critical levels play no part in the solar atmosphere where the magnetic field is nearly horizontal. However, their numerical solutions gave no indication as to the manner in which the addition of a small vertical field to a vertically sheared horizontal field possessing a critical level leads to the annihilation of that critical level. One of the purposes of the present study is to clarify the manner in which the vertical field suppresses the critical levels. This is carried out in §3 below and discussed in §5.

In the absence of a vertical field, it is known (Furth, Killeen & Rosenbluth 1963) that the addition of a small electrical resistivity r to the horizontal field introduces new features. Diffusion becomes potent in a thin region, whose thickness is defined in terms of r, situated on z_c. The introduction of resistivity not only smooths out the ideal-fluid solutions across the singularity but also adds two more localized solutions within the critical layer. The identification of these resistive instabilities depends on the solutions within the critical layer matching uniformly to the ideal solutions outside the layer where the effect of small resistivity is non-existent. This problem has been solved by Baldwin & Roberts (1972) using Laplace transform techniques. Since the addition of a small vertical field suppresses the critical level, it is of interest to examine the relative importance of resistivity and vertical field in order to determine whether the vertical field can suppress the resistive instabilities present in its absence.

One view of the Earth's magnetic field is that a large toroidal (horizontal) magnetic field is maintained against ohmic losses by a large (horizontal) differential rotation twisting a small poloidal (vertical) field to give rise to the so-called $\alpha\omega$-dynamo. The dynamo here is maintained by a balance being struck between magnetic, Coriolis and pressure forces. Although the ratio of horizontal to vertical fields is about 20, the dynamo will fail if the vertical field is absent. Now most stability studies of this so-called magnetostrophic-balance regime have neglected the poloidal field and critical layers were not present because of the simple horizontal field profiles used (see e.g. Eltayeb & Kumar 1977; Roberts & Loper 1979). Indeed critical layers are present in rotating fluids in situations of direct relevance to the dynamo problem when more general basic states are studied (Braginskii & Roberts 1975; Fearn 1984). Now the introduction of rotation to the horizontal magnetic field changes the position of the critical level (Eltayeb 1977, §3) and also influences the possible basic states. In the present study the role of rotation will not be considered. It is anticipated that the general conclusions reached may persist (qualitatively) in the presence of rotation. However, rotation may lead to new effects (see §5 below).

In §2, the problem is formulated, in §3 the influence of a small vertical field is studied in the absence of diffusion, in §4 the simultaneous action of small vertical field and small resistivity is examined and in §5 the results are discussed.

2. Formulation of the problem

Consider an inviscid fluid of electrical resistivity r permeated by a magnetic field $\boldsymbol{B}$. Take a Cartesian system of coordinates $O(x, y, z)$ in which Oz is vertically upwards and Ox, Oy horizontal. Suppose the fluid is contained between the two horizontal planes $z = z_1, z_2$. The velocity $\boldsymbol{u}$, pressure P and density ρ are related to $\boldsymbol{B}$ through the equations:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla P+\mu^{-1}\nabla\wedge\boldsymbol{B}\wedge\boldsymbol{B}+\rho\boldsymbol{g}, \tag{2.1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}(\boldsymbol{u}\wedge\boldsymbol{B})+\mu^{-1}r\nabla^2\boldsymbol{B}, \tag{2.2}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2.3}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.4}$$

$$\frac{\partial\rho}{\partial t}+\boldsymbol{u}\cdot\nabla\rho = 0, \tag{2.5}$$

in which $\boldsymbol{g}$ is the gravitational acceleration and μ the magnetic permeability.

Consider a basic state in which

$$\boldsymbol{u} = 0, \quad \boldsymbol{B} = \boldsymbol{B}_0 = (B_{\mathrm{H}}(z), 0, B_z), \quad P = P_0, \quad \rho = \rho_{00}\exp(-\beta z), \tag{2.6}$$

where B_z and ρ_{∞} are constants. The pressure P_0 is then governed by

$$\nabla\left(P_0+\frac{B_0^2}{2\mu}\right) = \mu^{-1}B_z\frac{\mathrm{d}}{\mathrm{d}z}B_{\mathrm{H}}\,\hat{\boldsymbol{x}}-\rho g\hat{\boldsymbol{z}}, \tag{2.7}$$

in which $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{z}}$ are unit vectors along Ox, Oz respectively. The absence of a basic flow and rotation makes the force balance, represented by (2.7), between pressure, magnetic and gravity forces only. The vertical (total) pressure gradient is balanced by gravity and the vertical field B_z introduces a horizontal pressure gradient. Even if resistivity is neglected and the last term of (2.2) is absent, (2.7) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}B_{\mathrm{H}}(z) = 0, \tag{2.8}$$

so that the horizontal field is linear in z. It is noteworthy that in the absence of B_z, the profile of $B_{\mathrm{H}}(z)$ is arbitrary.

Assume perturbations of the form

$$f(z)\exp\mathrm{i}(\omega t+kx), \tag{2.9}$$

in the velocity, magnetic field, pressure and density. Take a unit of length L, a unit of time L/V and a unit of velocity $r/L\mu$. Here L may be taken as the distance between the planes z_1, z_2 if z_2-z_1 is finite, otherwise L may be taken to be of the order of the lengthscale of variation of the horizontal magnetic field B_{H} or of the order of the horizontal wavelength of the waves. V is the horizontal component of the basic Alfvén velocity defined in (2.12) below. Adopting the Boussinesq approximation, which neglects variations except when they occur in the gravity term, the linearized perturbation equations can be reduced to the pair

$$\left(D^2-\alpha^2+\frac{G}{p^2}\right)W = p^{-2}(F-\mathrm{i}\epsilon^2 D)(D^2-\alpha^2)\phi, \tag{2.10}$$

$$(F-\mathrm{i}\epsilon^2 D)W = [p^{-2}Q(D^2-\alpha^2)-1]\phi, \tag{2.11}$$

in which

$$\left.\begin{aligned} &B_{\mathrm{H}} = (\mu\rho_{00})^{\frac{1}{2}}\, VF(z), \quad B_z = (\mu\rho_{00})^{\frac{1}{2}}\, U, \quad \phi = \mathrm{i}pSb_z, \\ &D \equiv \frac{\mathrm{d}}{\mathrm{d}z}, \quad b_z = \boldsymbol{b}\cdot\hat{\boldsymbol{z}}, \quad W = \boldsymbol{u}\cdot\hat{\boldsymbol{z}}, \quad \alpha = kL, \\ &\epsilon^2 = \frac{V}{U\alpha}, \quad p = \frac{\mathrm{i}\omega L}{\alpha V}, \quad S = \frac{\mu L V}{r}\left(=\frac{\tau_{\mathrm{r}}}{\tau_{\mathrm{a}}}\right), \\ &\tau_{\mathrm{r}} = \frac{\mu L^2}{r}, \quad \tau_{\mathrm{a}} = \frac{L}{V}, \quad Q = \frac{p}{S\alpha}, \quad G = -\beta g \tau_{\mathrm{a}}^2. \end{aligned}\right\} \tag{2.12}$$

Here τ_{a} is the Alfvénic timescale used as a unit of time, τ_{r} is the resistive timescale, S is the Lundquist number, G is a measure of the gravity effects and ϵ measures the effects of the vertical field. In view of (2.8) $F(z)$ takes the form $1+\gamma z$, for some γ, but we find it useful particularly in comparing the results with other situations in which $F'' \neq 0$ to use F and F' in the analysis below.

Elimination of W from (2.10) and (2.11) yields a single equation for ϕ

$$(\gamma_4 D^4 + \gamma_3 D^3 + \gamma_2 D^2 + \gamma_1 D + \gamma_0)\phi = 0, \tag{2.13}$$

in which

$$\left.\begin{aligned} &\gamma_4 = Q+\epsilon^4, \quad \gamma_3 = 2\mathrm{i}\epsilon^2 F - \frac{\gamma_4 H'}{H}, \\ &\gamma_2 = -\gamma_4\alpha^2 - p^2 + \mathrm{i}\epsilon^2 F' - F\left(F + \frac{\mathrm{i}\epsilon^2 H'}{H}\right) - Q\left(\alpha^2 - \frac{G}{p^2}\right) + 2Q\left[F'^2 - \mathrm{i}\epsilon^2 F'\left(\alpha^2 - \frac{G}{p^2}\right)\right]\Big/ H, \\ &\gamma_1 = \frac{(\gamma_4\alpha^2 + p^2)H'}{H} - 2\mathrm{i}\epsilon^2\alpha^2 F, \\ &\gamma_0 = -\mathrm{i}\epsilon^2\alpha^2 F' + \alpha^2 F\left(F + \frac{\mathrm{i}\epsilon^2 H'}{H}\right) - \frac{2F'(p^2+\alpha^2 Q)}{H} + \left(\alpha^2 - \frac{G}{p^2}\right)(p^2+\alpha^2 Q)\left(1 + \frac{2\mathrm{i}\epsilon^2 F'}{H}\right), \end{aligned}\right\} \tag{2.14}$$

where

$$H = F^2 + \mathrm{i}\epsilon^2 F' + \epsilon^4\left(\alpha^2 - \frac{G}{p^2}\right), \tag{2.15}$$

and the prime denotes differentiation with respect to the argument z.

When $\epsilon = 0$, and the vertical field is absent, (2.14) reduces to the equation studied by Baldwin & Roberts (1972). In that case $H = F^2$ and the equation becomes singular where F vanishes. When the vertical field is non-zero, H is non-zero at $F = 0$. However, if ϵ is very small then H is also very small near $F = 0$ and the problem requires detailed analysis in the neighbourhood of $F = 0$. It is this situation we intend to examine in order to identify the manner in which the vertical field suppresses the critical layer. For this purpose we shall assume that $0 < \epsilon \ll 1$.

In the absence of resistivity the governing equation can be obtained from (2.13) by formally setting $S = \infty$ (i.e. $Q = 0$). However, in this case it is found that a single equation for W is more convenient to deal with. It is

$$D[p^2 + (F - \mathrm{i}\epsilon^2 D)^2]\, DW - \alpha^2[p^2 + (F - \mathrm{i}\epsilon^2 D)^2]\, W + GW = 0. \tag{2.16}$$

The analysis given below showed that the presence of gravity complicates the solution (see Baldwin & Roberts 1972) and in the absence of G the solutions can be obtained using standard methods. However, for the main purpose of examining the influence of the vertical field, it can be argued that the presence of G will not change the qualitative nature of the results. When both Q and ϵ vanish (i.e. both resistivity and vertical field are absent) the wave equation takes the form

$$D^2[(p^2+F^2)DW]-\alpha^2(p^2+F^2)W = 0, \quad \phi = -FW, \tag{2.17}$$

which has singularities at

$$F(z_c) = \pm \mathrm{i}p. \tag{2.18}$$

These singularities lie on the real (z) axis if either $p = 0$ or $p = \mathrm{i}c$, c real. When resistivity is added, p is altered by an amount of the order of the thickness of the resistive layer (see §4 below) resulting in instability in certain circumstances. The cases of interest, in the context of the present basic state, are (i) $F = 0$ at z_c and (ii) $F = \pm c$, $F' \neq 0$ at $z = z_c$. We shall discuss these two cases in the absence and presence of diffusion in §§3 and 4 respectively.

In the neighbourhood of the critical layer z_c, the solution of (2.17) takes the form

$$W = A+\frac{1}{(z-z_c)} \quad \text{for } 0 < |p| \ll |z-z_c| \ll 1, \tag{2.19}$$

for case (i); and

$$W = A+\ln(z-z_c) \quad \text{for } 0 < |F\pm c| \ll |z-z_c| \ll 1, \tag{2.20}$$

for case (ii). In (2.20) the upper (lower) signs refer to the critical layer at $F = \mp c$.

Before we proceed to §3 we mention here that the solutions away from z_c are required to satisfy certain conditions which depend on the nature and position of z_1 and z_2. For finite values of z_1, z_2 the normal component of velocity, W, must vanish there if the planes are rigid, and is continuous for free boundaries. The magnetic boundary condition depends on the conductivity of the boundary and in the presence of the vertical field boundary layers are present (e.g. Eltayeb 1975). If either z_1 or z_2 or both lie at infinity then the radiation condition, which demands that waves propagating energy away from z_c are the only legitimate solutions, applies. Since our concern here is to identify the solutions across the critical layer, we will not be concerned with specific boundary conditions.

3. The diffusionless layer

In this section, we shall study the solution of (2.16) when $G = 0$ and $0 < \epsilon \ll 1$. It is clear that (2.16) reduces to (2.17) away from the critical level z_c for both cases (i) and (ii). The two solutions away from the critical level z_c behave as in (2.19), (2.20). The other two solutions are located in a thin magnetic layer situated on z_c. In order to ascertain the form of these solutions away from the critical level, z_c, we shall first examine the solution of (2.16) in the WKBJ approximation.

3.1. *WKBJ solutions*

Let

$$W = \exp\int^z g(z)\,\mathrm{d}z, \tag{3.1}$$

in which

$$g(z) = \epsilon^{-2}g_0(z)+\epsilon^0 g_1(z)+\epsilon^2 g_2(z)+\dots. \tag{3.2}$$

Substitute from (3.1) and (3.2) into (2.16), with $G = 0$, and equate the coefficients of ϵ^{2n} ($n = -1, 0, 2, \dots$) to zero. The coefficients of ϵ^{-2}, ϵ^0, ϵ^2 give either

$$g_0 = -\mathrm{i}F\pm p, \quad g_1 = \frac{\mathrm{i}F'}{-\mathrm{i}F\pm p}, \tag{3.3}$$

or $$g_0 = 0, \quad (p^2+F^2)(g_1'+g_1^2-\alpha^2)+2F'Fg_1 = 0. \tag{3.4}$$

Equation (3.4) is merely (2.17) and yields the solutions in the absence of the vertical field. The solutions (3.3), on the other hand, correspond to the two modes whose existence is due to the presence of the vertical field. Using the expressions for $F(z)$ near z_c, the solutions corresponding to (3.3) become

$$[F_c'(z-z_c)\pm ip]^{-1}\exp\left\{\frac{-i}{2\epsilon^2}[F_c'(z-z_c)\pm ip]^2\right\}, \tag{3.5}$$

for case (i); and

$$[F_c'(z-z_c)]^{-1}\exp\left\{-\frac{iF_c'}{2\epsilon^2}(z-z_c)^2\right\}, \quad (2c)^{-1}\exp\left(-\frac{2icz}{\epsilon^2}\right), \tag{3.6}$$

for case (ii). The subscript c refers to the critical level z_c. In (3.6) the solution is written down for $F = c$. If $F = -c$, we merely replace c by $-c$ in both solutions.

There are two points to note here. Firstly, the lengthscale of the magnetic layer is of order ϵ. Secondly, the two solutions in case (ii) are such that one of them is not influenced by the critical layer and varies on a *smaller* length scale of order ϵ^2. This last point can be clarified by appealing to the wave normal curves of (2.16). If we assume solutions of the form $\exp i(\omega t+kx+mz)$, and revert to dimensional quantities (c.f. (2.12) above), we obtain the *local* dispersion relation

$$(m^2+k^2)\{\omega^2-(k\bar{V}+mU)^2\} = N^2k^2, \quad N^2 = \beta g, \tag{3.7}$$

in which $\bar{V} = VF(z)$ and N is the Väisälä–Brunt frequency. For given ω and k there are four values of m corresponding to the four possible solutions. When $N = 0$, stratification is absent, the wave normal curves reduce to the two discontinuous straight lines described as asymptotes. The other two solutions are not present in the figure because they correspond to evanescent waves for which m $(=\pm ik)$ is purely imaginary. When $\epsilon \ll 1$, the two values of m for a given value of k (see ordinate CDE in figure 1) are such that one has a large magnitude and the other a small magnitude because the slope of the two lines is large and negative. The smaller of the two values of m corresponds to the first solution in (3.6) while the larger one corresponds to the second. Thus the solutions (3.6) represent two waves, one upgoing and the second downgoing.

The WKBJ solutions (3.5), (3.6) define the anti-Stokes lines for each case

$$\arg(z-z_c) = (0, \tfrac{1}{2}\pi, \pi, \tfrac{3}{2}\pi)-\tfrac{1}{2}\arg(F_c'). \tag{3.8}$$

These lines, in the complex z-plane, radiate from the point z_c. On each line the corresponding solution has an exponent with zero real part so that the nature of the solutions changes from a decaying one on one side to a growing one on the other side or vice versa. This property will be used frequently in the analysis below although the details are omitted. Refer to the book by Drazin & Reid (1981, §27) for the significance of anti-Stokes lines.

We now proceed to solve (2.16) in the magnetic layer.

3.2. *Case (i)*

In the neighbourhood of $z = z_c$, where $F(z_c) = 0$, we have

$$F = F_c'(z-z_c), \quad \zeta = \frac{F_c'(z-z_c)}{\delta_\epsilon}, \quad \delta_\epsilon = \epsilon|F_c'|^{\frac{1}{2}}, \quad p = A\delta_\epsilon, \tag{3.9}$$

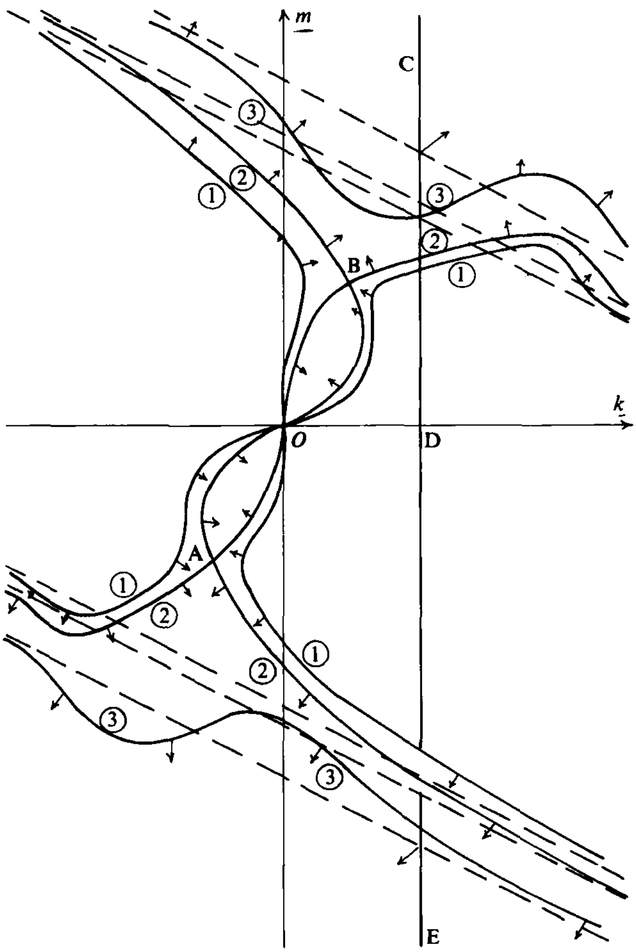

FIGURE 1. The wave normal curves for an inclined magnetic field. The discontinuous straight lines are the asymptotes $m = -(\bar{V}/U)\,k \pm (\omega^2 - N^2U^2)/(U^2+\bar{V}^2)^{\frac{1}{2}}$. The numbers indicate decreasing values of N^2. Curves ① are typical for $N^2 > \omega^2/V_s$; curves ② refer to $\omega^2 \leqslant N^2 \leqslant \omega^2/V_s^2$ and curves ③ correspond to $N^2 < \omega^2$. $V_s = \bar{V}/(\bar{V}^2+U^2)^{\frac{1}{2}}$. The points A and B are equidistant from the origin O, and they always lie on the line $m = kU/\bar{V}$. As N^2 decreases from values greater than ω^2/V_s, curves ① bend to touch at A and B when $N^2 = \omega^2/V_s^2$. As N^2 decreases further the bow-tie AOB turns slowly in the clockwise direction and both arms shrink gradually until $N^2 = \omega^2$ when they both disappear. Further increase in N^2 results in the open branches ③. In the meantime the asymptotes move steadily away from the axes. When N^2 approaches zero, the curves approach the pair of asymptotes, and the other two waves (strongly influenced by gravity) become evanescent. The small arrows indicate the direction of group velocity. The figure is drawn for $U^2 < \bar{V}^2$.

so that (2.16), when $G = 0$, reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}+\mathrm{i}\zeta-A\right]\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}+\mathrm{i}\zeta+A\right]\frac{\mathrm{d}W}{\mathrm{d}\zeta} = 0, \qquad (3.10)$$

where we have assumed $F_c' > 0$. If $F_c' < 0$, the two terms $\mathrm{i}\zeta - A$ and $(\mathrm{i}\zeta + A)$ are replaced by $-(\mathrm{i}\zeta - A)$ and $-(\mathrm{i}\zeta + A)$ respectively and the solutions (3.11)–(3.12) remain unchanged. The solution of (3.10) can be written as

$$W = D_0 + C_0\,u_1 + B_0\,u_2 + A_0\int_\infty^\zeta \left[\mathrm{e}^{-\frac{1}{2}\mathrm{i}\bar{\xi}_2^2}\int_{-\infty}^t \mathrm{e}^{\frac{1}{2}\mathrm{i}(\xi+\mathrm{i}A)^2}\,\mathrm{d}\xi - \mathrm{e}^{\frac{1}{2}\mathrm{i}\bar{\xi}_1^2}\int_\infty^t \mathrm{e}^{\frac{1}{2}\mathrm{i}(\xi-\mathrm{i}A)^2}\,\mathrm{d}\xi\right]\mathrm{d}t, \qquad (3.11)$$

in which

$$u_1 = \int_{-\infty}^{\zeta} e^{-\frac{1}{2}i(t-iA)^2}\,dt, \quad u_2 = \int_{\infty}^{\zeta} e^{-\frac{1}{2}i(t+iA)^2}\,dt, \tag{3.12}$$

and

$$\bar{\xi}_1 = t-iA, \quad \bar{\xi}_2 = t+iA. \tag{3.13}$$

The solution (3.11)–(3.14) has been written assuming $\mathrm{Re}(A) < 0$. If $\mathrm{Re}(A) > 0$, then the integrals in the square brackets in (3.11) and (3.12) have their $\pm\infty$ interchanged. The limits are decided by considerations of the anti-Stokes lines (3.8) so that

$$\Phi_1(\zeta) = e^{-\frac{1}{2}i(\zeta-iA)^2}, \quad \Phi_2(\zeta) = e^{-\frac{1}{2}i(\zeta+iA)^2}, \tag{3.14}$$

satisfy the relations

$$\Phi_1(\zeta e^{i\pi}) = \Phi_2(\zeta), \quad \Phi_2(\zeta e^{i\pi}) = \Phi_1(\zeta), \tag{3.15}$$

and both solutions are recessive. It will be shown presently that u_1, u_2 correspond to the WKBJ solutions (3.5) while the other two solutions match uniformly to the horizontal field solution given by (2.19). Indeed it is straightforward to show that

$$W \sim D_0 + \frac{iC_0}{\xi_1} e^{-\frac{1}{2}i\xi_1^2} + \frac{iB_0}{\xi_2} e^{-\frac{1}{2}i\xi_2^2} + iA_0 \ln\left(\frac{\xi_1}{\xi_2}\right) \quad (\zeta \to \infty). \tag{3.16}$$

and in view of (3.15),

$$W \sim D_0 + \frac{iC_0}{\xi_1} e^{-\frac{1}{2}i\xi_1^2} + \frac{iB_0}{\xi_2} e^{-\frac{1}{2}i\xi_2^2} + iA_0 \ln\left(\frac{\xi_1}{\xi_2}\right) \quad (\zeta \to -\infty), \tag{3.17}$$

where

$$\xi_1 = \zeta - iA, \quad \xi_2 = \zeta + iA, \tag{3.18}$$

and the natural logarithm is interpreted as containing the correct argument. Now

$$\ln\left(\frac{\xi_1}{\xi_2}\right) = \ln\left|\frac{\zeta-iA}{\zeta+iA}\right| + i(\arg\xi_1 - \arg\xi_2), \tag{3.19}$$

so that, assuming A real,

$$\ln\left(\frac{\xi_1}{\xi_2}\right) = \ln\left|\frac{\zeta-iA}{\zeta+iA}\right| - \frac{iA}{\zeta} - \frac{iA}{\zeta} \quad (\zeta \to +\infty), \tag{3.20}$$

$$\ln\left(\frac{\xi_1}{\xi_2}\right) = \ln\left|\frac{\zeta-iA}{\zeta+iA}\right| + i\left(\pi - \frac{A}{\zeta} + \pi - \frac{iA}{\zeta}\right) \quad (\zeta \to -\infty). \tag{3.21}$$

Moreover, comparison of (3.16), (3.17) with (3.5) shows that the coefficients of B_0 and C_0 are identical with the WKBJ solutions and the remainder of W, W_H, is

$$W_H = \begin{cases} D_0 + \dfrac{2A_0A}{\zeta} & (\zeta \to \infty), \\ D_0 - 2\pi A_0 + \dfrac{2A_0A}{\zeta} & (\zeta \to -\infty). \end{cases} \tag{3.22}$$

If we write (2.19) in the form

$$W = \begin{cases} A_+ + \dfrac{1}{(z-z_c)} & (z > z_c) \\ A_- + \dfrac{1}{(z_c-z)} & (z < z_c) \end{cases} \tag{3.23}$$

then (3.22) and (3.23) lead to the matching condition for W across the magnetic layer

$$W(z_{c+})-W(z_{c-}) = [W(z)] = \frac{\pi|F_c'|}{p}. \tag{3.24}$$

The jump $[W(z)]$ in W is merely (A_+-A_-) which can be obtained from knowledge of the solution away from the critical layer and the application of the appropriate boundary conditions at z_1, z_2 which determine A_+ and A_-. There is a variety of situations of specific examples that can be considered but it is not our intention to study these possibilities here. It may, however, be of interest to mention the stability of the situation in which the boundaries (assumed to lie at $z = \pm\frac{1}{2}$) are electrically perfectly conducting so that no boundary layers are invoked by the presence of the vertical field (see, for example, Eltayeb 1975). In this case we take $F = z$ so that (2.17) outside the critical layer at $z = 0$ possesses the solution

$$W_+ = \frac{\cosh \alpha z}{z} + A_+ \frac{\sinh \alpha z}{\alpha z},$$

$$W_- = \frac{\cosh \alpha z}{z} + A_- \frac{\sinh \alpha z}{\alpha z}.$$

Application of the jump condition (3.24) yields

$$p = -\tfrac{1}{4}\pi \frac{\tanh(\frac{1}{2}\alpha)}{\frac{1}{2}\alpha}$$

and the layer is stable.

3.3. *Case (ii)*

Here we set $p = \mathrm{i}c$, where c is predominantly real. We set

$$F = \pm(c+\mathrm{i}\delta_\epsilon A)+\delta_\epsilon \zeta, \quad \zeta = \frac{F_c'(z-z_c)}{\delta_\epsilon}, \quad \delta_\epsilon = \epsilon|F_c'|^{\frac{1}{2}}. \tag{3.25}$$

Equation (2.16) then becomes

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}+\mathrm{i}\zeta\mp A\right]\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\pm\frac{2\mathrm{i}c}{\delta_\epsilon}+\mathrm{i}\zeta\mp A\right]\frac{\mathrm{d}W}{\mathrm{d}\zeta} = 0, \tag{3.26}$$

in which the upper (lower) signs refer to the corresponding ones in the first of (3.25). Equation (3.6) is derived for $F_c' > 0$. If $F_c' < 0$, then $\mathrm{i}\zeta + A$ is replaced by $-(\mathrm{i}\zeta+A)$ and (3.29) and (3.30) below remain unchanged. Within the magnetic layer of thickness δ_ϵ this equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}+\mathrm{i}\zeta\mp A\right]\frac{\mathrm{d}W}{\mathrm{d}\zeta} = 0, \tag{3.27}$$

and the 'missing' fourth solution is given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\pm 2\mathrm{i}c\right)W = 0; \quad \xi = \delta_\epsilon^{-1}\zeta, \tag{3.28}$$

which is confined to an inner layer of thickness δ_ϵ^2. This solution clearly corresponds to the second WKBJ solution of (3.6).

The solution of (3.28) can simply be written as

$$W = C_0+B_0 u(\zeta)+A_0\int_{-\infty}^{\zeta} \mathrm{e}^{-\frac{1}{2}\mathrm{i}(t\pm \mathrm{i}A)^2}\,\mathrm{d}t\int_{\infty}^{t} \mathrm{e}^{\frac{1}{2}\mathrm{i}(\xi\pm A)^2}\,\mathrm{d}\xi, \tag{3.29}$$

in which

$$u(\zeta) = \int_{-\infty}^{\zeta} e^{-\frac{1}{2}i(t\pm iA)^2}\,dt. \tag{3.30}$$

The integrals in (3.29) and (3.30) assume $\mathrm{Re}(A) > 0$. If $\mathrm{Re}(A) < 0$ then the lower limits $\pm\infty$ should be interchanged. In this way the role played by the anti-Stokes lines is taken care of.

The solution $u(\zeta)$ corresponds to the WKBJ solution given by the first of (3.6) and the remainder of W matches uniformly to the horizontal field solution given by (2.20). Indeed it can be shown that

$$W \sim \begin{cases} C_0 + \dfrac{iB_0}{\zeta \pm iA}\, e^{-\frac{1}{2}i(\zeta\pm iA)^2} - iA_0 \ln \zeta & (\zeta \to \infty), \\ C_0 + \dfrac{iB_0}{\zeta \pm iA}\, e^{-\frac{1}{2}i(\zeta\pm iA)^2} \pm \pi A_0 - iA_0 \ln(-\zeta) & (\zeta \to -\infty), \end{cases} \tag{3.31}$$

and the matching of the solution

$$W = \begin{cases} A_+ + \ln(z - z_c) & (z > z_c), \\ A_- + \ln(z_c - z) & (z < z_c), \end{cases} \tag{3.32}$$

away from the layer is then

$$W(z_{c+}) - W(z_{c-}) = [W(z)] = \mp i\pi. \tag{3.33}$$

This condition shows that the jump in W across z_c depends on whether $F = c$ or $F = -c$ at z_c. Moreover, the total jump across a number of singularities will depend on their number as well as their type. In particular the result (3.33) is consistent with the analysis by Eltayeb (1981*b*). If there are two singularities one at $F = c$ and another at $F = -c$ then the net jump across both singularities vanishes.

The analysis presented in this section has shown that the presence of a small vertical magnetic field (in the sense that $\epsilon^2 \ll 1$) provides a mechanism of smoothing out the solutions in the critical layer present in its absence i.e. the solution can be uniformly connected across the level z_c without appealing to causality arguments (Miles 1961). The energy flux per unit mass of the system can be written as

$$\boldsymbol{F} = P\boldsymbol{u} + \tfrac{1}{2}u^2\boldsymbol{u} + V^2\boldsymbol{u} - (\boldsymbol{V}\cdot\boldsymbol{u})\,\boldsymbol{V}, \tag{3.34}$$

(cf. Eltayeb & McKenzie 1977) where $\boldsymbol{V}$ is the total (perturbation + basic) Alfvén velocity. The mean wave energy flux in the z-direction is the expression (3.34) averaged over a wavelength less the energy flux due to the basic state. In dimensional units this becomes

$$\bar{F}_{\mathrm{wave}} = \overline{\Pi W}, \tag{3.35}$$

where the bar denotes the average. By using the perturbation equations we can express (3.35) as

$$\bar{F}_{\mathrm{wave}} = \frac{V}{c}\{-(-c^2 + F^2)\,\overline{Wu} - \epsilon^2[2F\mathrm{i}\overline{Wu'} + F'\mathrm{i}\overline{Wu}] - \epsilon^4(\overline{Wu''})\}, \tag{3.36}$$

where

$$u = \boldsymbol{u}\cdot\hat{\boldsymbol{x}} = \frac{i}{\alpha}\, W'. \tag{3.37}$$

In the absence of the vertical field, $\epsilon = 0$ and the energy flux is proportional to the wave invariant of the system

$$\mathscr{A} = \left(-1 + \frac{F^2}{c^2}\right)\overline{Wu},$$

by
$$\bar{F}_{0\,\mathrm{wave}} = -Vc\mathscr{A},$$
so that $F_{0\,\mathrm{wave}}$ is constant everywhere except at critical levels where it experiences a finite jump (Eltayeb 1981*b*, §5). When $\epsilon \neq 0$, however, the mean wave energy flux in the vertical direction is not constant and the terms in (3.36) involving ϵ contribute to $\bar{F}_{\mathrm{wave}}$. Away from the critical level these terms are negligibly small but as the critical level is approached they become more important. If $F = 0$ at z_c then the terms reach their maximum order of magnitude of ϵ but for case (ii), where $F_c \neq 0$, the ϵ-terms become comparable with the first term. Moreover, the vertical field also influences the first term in (3.36) through its effect on the solutions W and u within the critical layer.

A hydromagnetic critical level is defined as a curve along which the phase speed of the wave matches the local Alfvén speed, and energy is forced to propagate along that field line (El Sawi & Eltayeb 1981). When a small vertical field of uniform value is added to a horizontal field sheared vertically the field lines become inclined to the horizontal and it becomes impossible for the horizontal phase speed of the wave to match the local Alfvén speed although it may be close to it if ϵ is small enough. The local Alfvén speed becomes inclined to the horizontal and its non-zero vertical component is responsible for the two new modes. If the vertical field is gradually increased the vertical component of the local Alfvén speed increases and consequently the lengthscale of the 'new' modes increases and their domain of influence spreads until it spans the whole region when $\epsilon = O(1)$ and the critical layer disappears completely.

The introduction of the vertical field then introduces two Alfvén waves, one downgoing and the other upgoing, which smooth out the solutions across the critical level. This mechanism is different from the influence of a small resistivity which is invoked by the shortening lengthscale of variation of the solutions as the critical level is approached. The solutions brought about by diffusion either grow or decay with time. Nevertheless, the simultaneous action of a small resistivity and a small vertical field has not been studied before and it is of interest to examine it here. This is carried out in the next section.

4. The diffusive layer in the presence of the vertical field

The critical layer in the absence of the vertical field was studied by Gibson & Kent (1971) when $G = 0$ and by Baldwin & Roberts (1972) for all values of G. The presence of G introduces more types of instabilities as compared with the tear mode instability which is relevant to the case $G = 0$. The tear mode instability is known to be important to the problem of plasma confinement and has attracted considerable attention (see e.g. Steinolfson & Van Hoven 1984). In the present analysis we shall study the situation when $G = 0$. However, for the purpose of identifying the influence of the vertical field we anticipate that the conclusions reached below (see §5) will apply to the slow interchange modes present for $G \neq 0$.

4.1. *Case* (*i*)

If we adopt the scaling appropriate to the critical layer in the absence of the field, namely
$$F = F_c'(z-z_c), \quad \zeta = \frac{z-z_c}{\delta_r}, \quad \delta_r = \left(\frac{p}{S\alpha F_c'^2}\right)^{\frac{1}{4}}, \quad A = \left(\frac{S\alpha p^3}{F_c'^2}\right)^{\frac{1}{2}}, \tag{4.1}$$

equations (2.13)–(2.15) yield

$$\sigma(\zeta^2+\tau)\frac{d^4\phi}{d\zeta^4}-2\sigma\zeta\frac{d^3\phi}{d\zeta^3}+[-A(\zeta^2+\tau)-(\zeta+\tau)^2+2-2\tau\zeta^2]\frac{d^2\phi}{d\zeta^2}$$
$$+2A\zeta\frac{d\phi}{d\zeta}-2(A+\tau)\phi=0, \quad (4.2)$$

in which $$\sigma = 1+R, \quad R=\left(\frac{\delta_\epsilon}{\delta_r}\right)^4, \quad \tau=\frac{iR^{\frac{1}{2}}}{F_c'^2}. \quad (4.3)$$

When $\epsilon = 0$ and the vertical field is absent, we recover the equation studied by Gibson & Kent (1971). The presence of the vertical field is manifest in (4.2) through the parameters R and τ. If R is small then the diffusive critical layer thickness δ_r is much larger than the thickness of the magnetic layer δ_ϵ. The ideal solution away from $\zeta = 0$ is then adjusted by the diffusive layer and the influence of the vertical field can only be felt in an inner region about $\zeta = 0$. But the solution of (4.2) is regular just outside the layer of thickness δ_ϵ and consequently the influence of the vertical field can only be felt at second order. Indeed to first order in $R^{\frac{1}{2}}$ the problem (4.2) can be reduced to the problem with $R = 0$ if we neglect R $(= (R^{\frac{1}{2}})^2)$ in σ and use $\xi = \zeta+\frac{1}{2}\tau$ in place of ζ as the independent variable. An analysis on the lines of that used by Gibson & Kent (1971) shows that the field does not influence the leading-order problem.

When $R \gg 1$, $\sigma \simeq R$ and $|\tau| \gg 1$ and the use of the independent variable $\xi = \zeta\delta_r/\delta_\epsilon$ reduces (4.2) to the one obtainable in the absence of diffusion, i.e. the solution obtained in §3.2. Diffusion is suppressed. This applies even if $G \neq 0$.

When $R = O(1)$, both influences are potent and (4.2) must be solved in its entirety. This can be achieved by the use of Laplace transforms or by numerical means. We will not attempt the solution of (4.2) for $R = O(1)$ here. However it is evidently clear from the analysis of the extreme cases $R \ll 1$ and $R \gg 1$ that the vertical field will suppress resistive modes if $R > R_c$, where $R_c = O(1)$. Translated into the original parameters, this means that

$$\frac{U}{V} \geqslant a\alpha(S\alpha)^{-\frac{2}{3}}, \quad (4.4)$$

in which a is a constant depending on the profile $F(z)$ of the horizontal field and on G.

The examination of case (ii) showed similar results and we shall therefore not present them here.

5. Concluding remarks

The investigations on the influence of a small magnetic field on the critical layers of the dissipationless horizontal magnetic shear showed that the vertical field introduces two Alfvén waves which interact with the two waves produced by the horizontal field within a thin 'magnetic layer' situated on the critical level z_c. As a consequence the full solution is regular everywhere. This occurs at all critical levels whether the Alfvén speed vanishes on z_c or not. Also, the second-order ordinary differential equation obtained for the horizontal-field case becomes of fourth order in the presence of the vertical field. It is then not possible to construct a wave invariant for the system (Eltayeb 1977, §2). Consequently the mean wave energy flux in the vertical direction (see §3 above) is in general a function of z, although it acquires constant values far away from the magnetic layer. As the layer is approached from one side, the mean energy flux varies continuously across the magnetic layer to a constant value on the far side of the critical level.

The influence of the vertical field on the resistive modes of the horizontal-field model was also studied. It is known that for a horizontal-field profile $F(z)$ there are three levels where localized resistive instabilities can occur when the electrical resistivity r is small. These occur where (i) $F = 0$, (ii) $F = c$ and $F'_c \neq 0$, (iii) $F = c$, $F'_c = 0$. For case (i) two types of instability are possible. The slow interchange modes for which $G \leqslant O(1)$ and $\alpha \leqslant O(1)$, a category that includes the tear mode for which $G = 0$. The effect of the vertical field on these instabilities was studied in detail in case (i) of §4 above to find that the vertical field suppresses them if

$$\epsilon^2 \left(= \frac{U}{V\alpha} \right) \geqslant a(S\alpha)^{-\frac{2}{3}}, \tag{5.1}$$

where a is a constant depending on G and the profile $F(z)$ of the horizontal field (see immediately above (4.4)). This condition applies both to the tearing mode (growth rate $O((S\alpha)^{-\frac{3}{5}})$ and the slow gravitational interchange mode (growth rate $O((S\alpha)^{-\frac{1}{3}})$) because the thickness of the resistive layer is the same in both cases.

Another type of instability associated with case (i) is the so-called localized gravitational modes which occur for large values of α and have been shown by Baldwin & Roberts (1972, §6) to have growth rates

$$p = \frac{G^{\frac{1}{4}}}{\alpha} \left(1 - \frac{\frac{1}{2}\lambda F'_c S^{\frac{1}{2}}}{\alpha G^{\frac{1}{4}}} \right),$$

in which λ is an integer. A detailed analysis of the influence of the vertical field on this instability (omitted here for brevity) showed that they are suppressed if

$$\epsilon^2 \geqslant b S^{-\frac{1}{2}} \alpha^{-1}, \tag{5.2}$$

where b depends on F'_c. Investigations of the resistive modes for case (ii) showed results similar to (5.1).

Case (iii) lies outside the basic state of §2 above because $F''(z)$ vanishes everywhere and therefore $F(z)$ has no extreme values. Attempts to find a *simple* model for case (iii) failed. Even if uniform vertical rotation with a flow $U(z)$ parallel to the horizontal field were introduced we find that $F''(z) = 0$. It transpires that the basic state must depend on at least two independent variables or $U(z)$ be inclined to the field to maintain a non-vanishing $F''(z)$. A basic state of this type is outside the scope of the present study. If $F''(z)$ is maintained by some artificial source, however, a detailed analysis shows that the vertical field again suppresses the resistive instabilities if $\epsilon^2 \geqslant O((S\alpha)^{-\frac{3}{4}})$, the difference here being due to a magnetic layer of thickness $O(\epsilon^{\frac{2}{3}})$.

Part of this work was carried out at the International Centre for Theoretical Physics, Trieste, Italy and both authors wish to thank the Director, Professor Abdus Salam, for his hospitality. I. A. E. wishes to thank Professor D. Loper for making his visit to the Florida State University possible and also for useful discussions. This research was partially supported by the Earth Science Section of the National Science Foundation under grant EAR 840919 and is contribution number 229 of the Geophysical Fluid Dynamics Institute, Florida State University.

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